

# REPRESENTABILITY OF HILBERT SCHEMES AND HILBERT STACKS OF POINTS

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**ABSTRACT.** We show that the Hilbert functor of points on an arbitrary separated algebraic space is representable. We also show that the Hilbert stack of points on an arbitrary algebraic space or an arbitrary algebraic stack is algebraic.

## INTRODUCTION

The purpose of this note is to give a short and elementary proof of the existence of the Hilbert scheme of points  $\mathrm{Hilb}^d(X/S)$  for an arbitrary separated algebraic space  $X/S$ . Taking an étale cover  $f : U \rightarrow X$  we use the fact that there is an open subset  $\mathrm{reg}(f)$  of  $\mathrm{Hilb}^d(U/S)$  and an étale cover  $f_* : \mathrm{Hilb}^d(U/S)|_{\mathrm{reg}(f)} \rightarrow \mathrm{Hilb}^d(X/S)$ . In this way we reduce the representability to the affine case.

When  $X/S$  is not separated, the Hilbert functor is not representable [LS08]. A replacement for the Hilbert functor is then the Hilbert stack [Art74, Appendix]. With the same method as for the Hilbert functor, we can deduce the algebraicity of the Hilbert stack from the affine case. We also show that the open substack parameterizing étale families coincides with the stack quotient of  $(X/S)^d$  by the symmetric group. Finally we make some remarks on the generalization to the case where  $X$  is an algebraic stack. An application is the existence of the Weil restriction along a finite flat morphism for arbitrary algebraic spaces and algebraic stacks.

## 1. THE HILBERT FUNCTOR AND THE HILBERT STACK

**Definition (1.1).** We say that  $f : X \rightarrow Y$  is finite and flat of rank  $d$  if  $f$  is finite and  $f_*\mathcal{O}_X$  is a locally free  $\mathcal{O}_Y$ -module of constant rank  $d$ .

**Definition (1.2).** Let  $X/S$  be a *separated* scheme (resp. separated algebraic space). The Hilbert functor of points  $\mathcal{Hilb}_{X/S}^d$  is the functor which to an  $S$ -scheme  $T$  assigns the set of closed subschemes (resp. subspaces)  $Z \hookrightarrow X \times_S T$  such that the second projection  $p : Z \rightarrow T$  is finite flat of rank  $d$ .

**Remark (1.3).** It is easily seen, using [EGA<sub>IV</sub>, Thm. 12.2.1 (i), (ii)], that  $\mathcal{Hilb}_{X/S}^d$  is an open and closed subfunctor of the Hilbert functor  $\mathcal{Hilb}_{X/S}$  which parameterizes closed subspaces which are flat, proper and of finite presentation.

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*Date:* 2008-02-26.

*2000 Mathematics Subject Classification.* Primary 14C05; Secondary 14A20.

*Key words and phrases.* Hilbert scheme, Hilbert stack, Weil restriction.

**Remark (1.4).** In [FGA, No. 221] Grothendieck shows that  $\mathcal{Hilb}_{X/S}$  is represented by the Hilbert scheme  $\mathrm{Hilb}(X/S)$  when  $X/S$  is quasi-projective and of finite presentation. Using Artin's criteria for representability it can be shown that  $\mathcal{Hilb}_{X/S}$  is represented by a separated algebraic space when  $X/S$  is a separated algebraic space, locally of finite presentation [Art69, Cor. 6.2]. It is further known that when  $X/S$  is affine, then  $\mathcal{Hilb}_{X/S}^d$  is represented by a scheme  $\mathrm{Hilb}^d(X/S)$  [Nor78, GLS07]. This also follows independently from Theorem (4.2).

When  $X/S$  is not separated then  $\mathcal{Hilb}_{X/S}$  and  $\mathcal{Hilb}_{X/S}^d$  are not representable [LS08]. If we want to study families, then the Hilbert stack [Art74, Appendix] is a replacement for the Hilbert functor. The difference between the Hilbert stack and the Hilbert functor is that in the stack we consider flat families  $Z \rightarrow T$  with morphisms  $Z \rightarrow X$  without the condition that  $Z \rightarrow X \times_S T$  is a closed immersion.

**Definition (1.5).** Let  $X$  be an algebraic space over a base scheme  $S$ . Let  $\mathcal{H}_X^d$  be the category where the objects are pairs of morphisms  $(p : Z \rightarrow T, q : Z \rightarrow X)$  where  $T$  is an  $S$ -scheme and  $p$  is finite and flat of rank  $d$ . The morphisms are cartesian diagrams

$$(1.5.1) \quad \begin{array}{ccc} Z_1 & \xrightarrow{\varphi} & Z_2 \\ \downarrow p_1 & \square & \downarrow p_2 \\ T_1 & \xrightarrow{\psi} & T_2 \end{array}$$

such that  $q_1 = q_2 \circ \varphi$ . Clearly  $\mathcal{H}_X^d$  is a category fibered in groupoids and it follows from étale descent of affine schemes [SGA<sub>1</sub>, Exp. VIII, Thm. 2.1] that  $\mathcal{H}_X^d$  is a stack. We will call  $\mathcal{H}_X^d$  the *Hilbert stack* of  $d$  points on  $X$ .

**Remark (1.6).** As for the Hilbert functor, it is clear that  $\mathcal{H}_X^d$  is an open and closed substack of the Hilbert stack  $\mathcal{H}_X$  which parameterizes flat, proper algebraic spaces  $Z$  of finite presentation with a morphism to  $X$ . To obtain algebraicity for  $\mathcal{H}_X$ , as  $Z/T$  in general is not projective, it is usual to require that  $(q, p) : Z \rightarrow X \times_S T$  is quasi-finite. This is always the case for  $\mathcal{H}_X^d$ . The algebraicity of  $\mathcal{H}_X$  for a *separated* algebraic space  $X$ , locally of finite presentation over  $S$ , is proved in [Sta06].

**Remark (1.7).** In [Vis91] Vistoli considers a variant of the Hilbert stack where  $(q, p) : Z \rightarrow X \times_S T$  is required to be unramified. Vistoli's Hilbert stack is an open substack of the Hilbert stack  $\mathcal{H}_X$  as is easily seen considering the relative cotangent sheaf  $\Omega_{Z/X \times_S T}^1$ .

When  $X$  is *separated*, then the Hilbert functor  $\mathcal{Hilb}_X^d$  is an open subfunctor of the Hilbert stack  $\mathcal{H}_{X/S}^d$ . In fact, when  $X/S$  is separated then  $(q, p)$  is finite, by Zariski's main theorem, and it follows from Nakayama's lemma that  $(q, p)$  is a closed immersion over an open subset of  $X \times_S T$ .

## 2. PUSH-FORWARD AND WEIL RESTRICTION

**Definition (2.1).** Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces. There is then a natural morphism  $f_* : \mathcal{H}_X^d \rightarrow \mathcal{H}_Y^d$  taking an object  $(p :$

$Z \rightarrow T, q : Z \rightarrow X$ ) to  $(p, f \circ q)$ . When  $Y$  is separated, we let  $\mathcal{H}ilb_{X, \text{reg}/f}^d = f_*^{-1}(\mathcal{H}ilb_Y^d)$ .

The stack  $\mathcal{H}ilb_{X, \text{reg}/f}^d$  is an open substack of  $\mathcal{H}_X^d$  without non-trivial automorphisms, i.e., it is a sheaf. In fact, if  $(p, q)$  is an object in  $\mathcal{H}ilb_{X, \text{reg}/f}^d$ , then  $q$  is a monomorphism as  $f \circ q$  is a closed immersion. When in addition  $X$  is separated then  $\mathcal{H}ilb_{X, \text{reg}/f}^d$  is an open subfunctor of  $\mathcal{H}ilb_X^d$ .

**Definition (2.2).** Let  $X' \rightarrow S'$  and  $S' \rightarrow S$  be morphism of algebraic spaces. The *Weil restriction*  $\mathbf{R}_{S'/S}(X')$  is the functor from  $S$ -schemes to sets that takes an  $S$ -scheme  $T$  to the set of sections of  $X'_T \rightarrow S'_T$ , i.e.,

$$\mathbf{R}_{S'/S}(X')(T) = \text{Hom}_{S' \times_S T}(S' \times_S T, X' \times_S T).$$

The Weil restriction is also sometimes denoted by  $\Pi_{X'/S'/S}$ , cf. [FGA, No. 195, §C 2], and also known as *restriction of scalars*.

**Remark (2.3).** Let  $S$  be a scheme and let  $S' \rightarrow S$  be a finite flat morphism of rank  $d$ . Let  $f : X' \rightarrow S'$  be an algebraic space. Then there is a natural isomorphism  $\mathbf{R}_{S'/S}(X') \cong \mathcal{H}ilb_{X', \text{reg}/f}^d$ . Note that  $\mathcal{H}ilb_{S'/S}^d \rightarrow S$  is an isomorphism. We thus have the following cartesian diagram

$$(2.3.1) \quad \begin{array}{ccc} \mathbf{R}_{S'/S}(X') & \longrightarrow & S \\ \downarrow & \square & \downarrow \\ \mathcal{H}_{X'/S}^d & \xrightarrow{f_*} & \mathcal{H}_{S'/S}^d \end{array}$$

where the two vertical morphisms are open immersions.

**Proposition (2.4).** Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces. If  $f$  is

- (i) *locally of finite presentation*
- (ii) *formally étale*
- (iii) *formally unramified*
- (iv) *formally smooth*
- (v) *surjective and smooth*
- (vi) *a closed immersion*
- (vii) *an open immersion*
- (viii) *affine*
- (ix) *quasi-affine*

then so is  $f_* : \mathcal{H}_X^d \rightarrow \mathcal{H}_Y^d$ . Furthermore, assuming that  $\mathcal{H}_X^d$  and  $\mathcal{H}_Y^d$  are algebraic stacks, then if  $f$  is

- (x) *locally of finite type*
- (xi) *quasi-compact*

then so is  $f_*$ .

*Proof.* Let  $P$  be one of the properties (i)-(xi) and assume that  $f$  has  $P$ . Let  $T$  be an affine scheme and  $T \rightarrow \mathcal{H}_Y^d$  be a morphism. It is enough to verify that  $\mathcal{H}_X^d \times_{\mathcal{H}_Y^d} T \rightarrow T$  has property  $P$ . Let  $p : Z \rightarrow T$  and  $q : Z \rightarrow Y$  be the morphisms corresponding to  $T \rightarrow \mathcal{H}_Y^d$ . It is then readily verified that

$\mathcal{H}_X^d \times_{\mathcal{H}_Y^d} T = \mathbf{R}_{Z/T}(X \times_Y Z)$ . We may thus also assume that  $Y = Z$  is affine.

Properties (i)-(v) are verified using the functorial characterization of morphisms which are locally of finite presentation [EGA<sub>IV</sub>, Prop. 8.14.2] and the infinitesimal criteria for formally étale, unramified and smooth maps. Property (vi) follows from [EGA<sub>I</sub>, Lem. 9.7.9.1] and (vii) is easy. For (viii) it is by (vi) enough to show that  $\mathbf{R}_{Z/T}(W)$  is represented by a scheme affine over  $T$  when  $W$  is the spectrum of a polynomial ring over  $\mathcal{O}_Z$ . This is straight-forward. We refer to [BLR90, §7.6, Prop. 2, pf. of Thm. 4, Prop. 5] for details.

To show (x) we take a smooth surjective cover  $U \rightarrow X$  such that  $U$  is a disjoint union of affine schemes. Then if  $X \rightarrow Y$  is locally of finite type, we can factor  $U \rightarrow Y$  through a closed immersion  $U \hookrightarrow W$  and a morphism  $W \rightarrow Y$  which is locally of finite presentation. Thus by (vi), (i) and (v), it follows that  $\mathcal{H}_X^d \rightarrow \mathcal{H}_Y^d$  is locally of finite type.

Similarly, to show (xi) we take a smooth cover with  $U$  affine and the quasi-compactness of  $f_*$  follows from (viii). Finally (ix) follows from (vii), (viii) and (xi).  $\square$

**Corollary (2.5).** *Let  $S$  be a scheme and let  $S' \rightarrow S$  be a finite flat morphism of rank  $d$ . Let  $f : X' \rightarrow S'$  be an algebraic space. If  $f$  has one of the properties in Proposition (2.4), then so has  $\mathbf{R}_{S'/S}(X') \rightarrow S$ , assuming that this morphism is representable for properties (x) and (xi).*

*Proof.* Follows from Proposition (2.4) and the diagram (2.3.1).  $\square$

**Remark (2.6).** Over families  $T \rightarrow \mathcal{H}_Y^d$  such that  $Z \rightarrow T$  is étale, (resp. étale morphisms  $S' \rightarrow S$ ) Proposition (2.4) (resp. Corollary (2.5)) also holds for the properties proper and flat [BLR90, Prop. 5].

**Example (2.7).** Proposition (2.4) does not hold for the property proper nor for the property finite and étale. In fact, let  $T$  be arbitrary and  $Z \rightarrow T$  be a finite flat ramified cover of degree  $d$ . Then  $\mathbf{R}_{Z/T}(Z \amalg Z) \rightarrow T$  is étale of generic rank  $2^d$  but has lower rank over the branch locus of  $Z \rightarrow T$ . Thus  $\mathbf{R}_{Z/T}(Z \amalg Z) \rightarrow T$  is not proper.

### 3. REPRESENTABILITY OF THE HILBERT SCHEME

**Definition (3.1).** Let  $X$  be a scheme. We say that  $X$  is an AF-scheme if every finite subset of points  $Z \subseteq X$  is contained in an affine open subset of  $X$ .

**Remark (3.2).** If  $S$  is an affine scheme and  $X \rightarrow S$  is a locally quasi-finite and separated morphism of algebraic spaces, then it follows from Zariski's main theorem [LMB00, Thm. A.2] that every finite subset  $Z \subseteq X$  is contained in a quasi-affine open subscheme of  $X$ . It then follows from [EGA<sub>II</sub>, Cor. 4.5.4] that  $X$  is AF.

**Proposition (3.3).** *Let  $S$  be an affine scheme and  $X/S$  an AF-scheme. Then  $\mathrm{Hilb}_{X/S}^d$  is represented by a scheme  $\mathrm{Hilb}^d(X/S)$ .*

*Proof.* Let  $X = \bigcup_{\alpha} U_{\alpha}$  be an open cover of  $X$  by affines such that every subset of  $d$  points of  $X$  lies in some  $U_{\alpha}$ . It is then easily seen that  $\coprod_{\alpha} \mathcal{Hilb}_{U_{\alpha}}^d \rightarrow \mathcal{Hilb}_X^d$  is a Zariski covering. As  $\mathcal{Hilb}_{U_{\alpha}}^d$  is represented by a scheme, cf. Remark (1.4), so is  $\mathcal{Hilb}_X^d$ .  $\square$

**Corollary (3.4).** *Let  $S$  be a scheme and let  $S' \rightarrow S$  be a finite flat morphism of rank  $d$ . Let  $X' \rightarrow S'$  be a locally quasi-finite and separated morphism. Then the Weil restriction  $\mathbf{R}_{S'/S}(X')$  is represented by a scheme.*

*Proof.* The question is local on  $S$  and we can thus assume that  $S$  is affine. Then  $X'$  is an AF-scheme and  $\mathcal{Hilb}_{X'}^d$  is represented by the scheme  $\mathrm{Hilb}^d(X')$ . By Remark (2.3), the Weil restriction  $\mathbf{R}_{S'/S}(X')$  is an open subscheme of  $\mathrm{Hilb}^d(X')$  and hence a scheme. A more direct proof is given in [BLR90, Thm. 4].  $\square$

We obtain the following complement to Proposition (2.4).

**Corollary (3.5).** *Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces. If  $f$  is locally quasi-finite and separated, then  $f_* : \mathcal{H}_X^d \rightarrow \mathcal{H}_Y^d$  is represented by schemes.*

**Theorem (3.6).** *Let  $X/S$  be a separated algebraic space. Then  $\mathcal{Hilb}_{X/S}^d$  is represented by a separated algebraic space which we denote  $\mathrm{Hilb}^d(X/S)$ .*

*Proof.* We can assume that  $S$  is affine. Let  $f : U = \coprod_{\alpha} U_{\alpha} \rightarrow X$  be an étale cover such that  $U_{\alpha}$  is affine. Then  $U$  is an AF-scheme and  $\mathcal{Hilb}_{U/S}^d$  is represented by the scheme  $\mathrm{Hilb}^d(U/S)$ . As  $f$  is separated, étale, surjective and locally quasi-finite, the morphism  $f_* : \mathcal{Hilb}_{U/S, \mathrm{reg}/f}^d \rightarrow \mathcal{Hilb}_{X/S}^d$  is étale, surjective and representable by Proposition (2.4) and Corollary (3.5). Thus  $\mathcal{Hilb}_{X/S}^d$  is an algebraic space.

That the diagonal of  $\mathcal{Hilb}_{X/S}^d \rightarrow S$  is a closed immersion follows easily from [EGA<sub>I</sub>, Lem. 9.7.9.1].  $\square$

**Corollary (3.7).** *Let  $S' \rightarrow S$  be a finite flat morphism of rank  $d$ . Let  $X'/S'$  be a separated algebraic space. Then the Weil restriction  $\mathbf{R}_{S'/S}(X')$  is represented by a separated algebraic space.*

*Proof.* By Remark (2.3), we have that  $\mathbf{R}_{S'/S}(X') = \mathrm{Hilb}_{X', \mathrm{reg}/S'}^d$  is an open subfunctor of  $\mathrm{Hilb}^d(X'/S)$  and is thus a separated algebraic space by Theorem (3.6).  $\square$

**Corollary (3.8).** *Let  $f : X \rightarrow Y$  be a separated morphism of algebraic spaces. Then  $f_* : \mathcal{H}_X^d \rightarrow \mathcal{H}_Y^d$  is represented by separated algebraic spaces.*

*Proof.* Follows from Corollary (3.7) as in the proof of Proposition (2.4).  $\square$

#### 4. ALGEBRAICITY OF THE HILBERT STACK

As for the Hilbert functor, the algebraicity of the Hilbert stack will be an immediate consequence of Proposition (2.4) after we have verified that the Hilbert stack of an affine scheme is algebraic in Theorem (4.2).

In the finitely presented case, the following results follows from the more general results of [Lie06, §2.1]. In the affine case treated below, the proof is a matter of elementary algebra.

**Lemma (4.1).** *Let  $B$  be an  $A$ -algebra and let  $M$  be a locally free  $A$ -module of finite rank. Then there is an  $A$ -algebra  $Q$  which represents  $B$ -algebra structures on  $M$ . That is, for every  $A$ -algebra  $A'$ , there is a functorial one-to-one correspondence between  $B' = B \otimes_A A'$ -algebra structures on  $M' = M \otimes_A A'$  and homomorphisms  $Q \rightarrow A'$ . If  $B$  is an  $A$ -algebra of finite type (resp. of finite presentation), then so is  $Q$ .*

*Proof.* A  $B'$ -algebra structure on  $M'$  is given by multiplication maps  $\mu : M' \otimes_{A'} M' \rightarrow M'$ ,  $m : B' \otimes_{A'} M' \rightarrow M'$ , and a unit  $\eta : A' \rightarrow M'$ . Such triples of maps correspond to  $A$ -module homomorphisms

$$(M \otimes_A M \otimes_A M^\vee) \oplus (B \otimes_A M \otimes_A M^\vee) \oplus M^\vee \rightarrow A'$$

and are thus represented by the symmetric algebra

$$P := \text{Sym}((M \otimes_A M \otimes_A M^\vee) \oplus (B \otimes_A M \otimes_A M^\vee) \oplus M^\vee).$$

That the multiplication  $\mu$  is commutative, associative and compatible with  $m$  and  $\eta$  can be expressed as the vanishing of the  $A'$ -homomorphisms

$$\begin{aligned} \mu - \mu \circ \tau &: M' \otimes_{A'} M' \rightarrow M' \\ \mu \circ (\mu \otimes \text{id}_M) - \mu \circ (\text{id}_M \otimes \mu) &: M' \otimes_{A'} M' \otimes_{A'} M' \rightarrow M' \\ \mu \circ (m \otimes \text{id}_M) - m \circ (\text{id}_B \otimes \mu) &: B' \otimes_{A'} M' \otimes_{A'} M' \rightarrow M' \\ \mu \circ (\eta \otimes \text{id}_M) - \text{id}_M &: M' \rightarrow M' \end{aligned}$$

where  $\tau : M' \otimes_{A'} M' \rightarrow M' \otimes_{A'} M'$  swaps the two factors. This vanishing is represented by a quotient  $Q$  of  $P$  according to [EGA<sub>I</sub>, Lem. 9.7.9.1].

If  $B$  is of finite type, then clearly so is  $P$  and hence  $Q$ . If  $B$  is of finite presentation we use a limit argument to reduce to the noetherian case and it follows that  $Q$  is of finite presentation.  $\square$

**Theorem (4.2).** *Let  $X$  and  $S$  be affine schemes. Then  $\mathcal{H}_{X/S}^d$  is a quasi-compact algebraic stack with affine diagonal. If  $X/S$  is of finite type (resp. of finite presentation) then so is  $\mathcal{H}_{X/S}^d$ .*

*Proof.* There is a natural morphism  $\mathcal{H}_{X/S}^d \rightarrow BGL_d(S)$  which maps  $(Z, p, q)$  to the locally free  $\mathcal{O}_S$ -module  $p_* \mathcal{O}_Z$ . The stack  $BGL_d(S)/S$  is a finitely presented algebraic stack with affine diagonal. Lemma (4.1) shows that  $\mathcal{H}_{X/S}^d \rightarrow BGL_d(S)$  is represented by affine morphisms and the theorem follows.  $\square$

**Remark (4.3).** Consider the following properties of a map  $f : X \rightarrow Y$ .

- (D) a closed immersion, unramified, affine, quasi-affine,  
quasi-compact, separated

The above properties are stable under base change, compositions and under compositions with closed immersions. Moreover, if  $f : X \rightarrow Y$  has one of the above properties, then so has  $f_* : \mathcal{H}_X^d \rightarrow \mathcal{H}_Y^d$  by Proposition (2.4) and Corollary (3.8). Similarly, if  $f : X' \rightarrow S'$  has one of the properties

in (D) and  $g : S' \rightarrow S$  is finite flat of rank  $d$ , then  $\mathbf{R}_{S'/S}(X') \rightarrow S$  has that property by Remark (2.3).

**Proposition (4.4).** *Let  $T \rightarrow \mathcal{H}_X^d \times_S \mathcal{H}_X^d$  be a morphism corresponding to the families  $(Z_1, p_1, q_1)$  and  $(Z_2, p_2, q_2)$ . Let  $\mathcal{I}som(Z_1, Z_2)$  be the pull-back  $\mathcal{H}_X^d \times_{\mathcal{H}_X^d \times_S \mathcal{H}_X^d} T$ . Then  $\mathcal{I}som(Z_1, Z_2)$  is a closed subscheme of*

$$\mathbf{R}_{Z_1/T}(Z_1 \times_{X \times_S T} Z_2) \times_T \mathbf{R}_{Z_2/T}(Z_1 \times_{X \times_S T} Z_2).$$

*In particular, if the maps  $(q_i, p_i) : Z_i \rightarrow X \times_S T$ ,  $i = 1, 2$  both have one of the properties in (D) then so has  $\mathcal{I}som(Z_1, Z_2) \rightarrow T$ .*

*Proof.* This follows easily from [EGA<sub>I</sub>, Lem. 9.7.9.1].  $\square$

**Theorem (4.5).** *Let  $X/S$  be an algebraic space. Then the stack  $\mathcal{H}_{X/S}^d$  is algebraic with separated diagonal. If  $X/S$  is separated (resp. quasi-separated), then  $\mathcal{H}_{X/S}^d$  has affine (resp. quasi-affine) diagonal. If  $X/S$  has one of the properties: quasi-compact, (locally) of finite type, (locally) of finite presentation; then so has  $\mathcal{H}_{X/S}^d/S$ .*

*Proof.* We can assume that  $S$  is affine. Let  $U = \coprod U_\alpha \rightarrow X$  be an étale cover such that  $U_\alpha$  is affine and any set of at most  $d$  points of  $X$  can be lifted to some  $U_\alpha$ . Equivalently, we require that  $\coprod_\alpha (U_\alpha)^d \rightarrow X^d$  is surjective. As  $\mathcal{H}_{U_\alpha/S}^d$  is algebraic by Theorem (4.2), there is a smooth presentation  $V_\alpha \rightarrow \mathcal{H}_{U_\alpha/S}^d$  with  $V_\alpha$  affine. The composition

$$\coprod_\alpha V_\alpha \rightarrow \coprod_\alpha \mathcal{H}_{U_\alpha/S}^d \rightarrow \mathcal{H}_{X/S}^d$$

is a smooth presentation of  $\mathcal{H}_{X/S}^d$  according to Proposition (2.4). The representability and the properties of the diagonal of  $\mathcal{H}_{X/S}^d$  then follows from Proposition (4.4). In fact, if  $X/S$  is separated (resp. quasi-separated) and  $(Z, p, q) \in \mathcal{H}_X^d(T)$ , then  $(q, p) : Z \rightarrow X \times_S T$  is finite (resp. quasi-finite and separated) and hence affine (resp. quasi-affine) by Zariski's Main Theorem [EGA<sub>IV</sub>, Prop. 18.12.12]. If  $X/S$  is quasi-compact then  $(X/S)^d$  is quasi-compact and a finite number of the  $(U_\alpha)^d$ 's cover  $(X/S)^d$  and it follows that  $\mathcal{H}_{X/S}^d$  is quasi-compact.  $\square$

**Corollary (4.6).** *Let  $S' \rightarrow S$  be a finite flat morphism of rank  $d$ . Let  $X'/S'$  be a, not necessarily separated, algebraic space. Then the Weil restriction  $\mathbf{R}_{S'/S}(X')$  is represented by an algebraic space.*

*Proof.* By Remark (2.3), we have that  $\mathbf{R}_{S'/S}(X') = \mathrm{Hilb}_{X', \mathrm{reg}/S'}^d$  is an open subfunctor of  $\mathcal{H}_{X'/S}^d$  and is thus an algebraic space.  $\square$

**Corollary (4.7).** *Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces. Then  $f_* : \mathcal{H}_X^d \rightarrow \mathcal{H}_Y^d$  is represented by algebraic spaces.*

*Proof.* Follows from Corollary (4.6) as in the proof of Proposition (2.4).  $\square$

## 5. ÉTALE FAMILIES

The *stack of branchvarieties* [AK06] is the open substack of the Hilbert stack  $\mathcal{H}_{X/S}$  parameterizing families  $(p : Z \rightarrow T, q : Z \rightarrow X)$  such that the geometric fibers of  $p$  are reduced. For zero-dimensional families of rank  $d$ , this is the open substack  $\mathcal{E}t_{X/S}^d$  of  $\mathcal{H}_{X/S}^d$  parameterizing étale families of rank  $d$ . If  $X/S$  is separated, it is natural to also study the subspace  $\text{ET}_{X/S}^d$  parameterizing étale families  $Z \rightarrow T$  of rank  $d$  such that  $Z \rightarrow X \times_S T$  is a closed immersion. Following the notation of [LMB00, 6.6] we let  $\text{SEC}_{X/S}^d$  be the open subset of  $(X/S)^d = X \times_S \cdots \times_S X$  which is the complement of the diagonals. The symmetric group  $\mathfrak{S}_d$  acts by permutations on  $(X/S)^d$  and this action is free over  $\text{SEC}_{X/S}^d$ . We have the following easy descriptions of these stacks:

**Theorem (5.1).** *Let  $X/S$  be an algebraic space. There is a canonical isomorphism  $\mathcal{E}t_{X/S}^d \rightarrow [(X/S)^d/\mathfrak{S}_d]$ . If  $X/S$  is separated, then the open substack  $\text{ET}_{X/S}^d$  is identified with the algebraic space  $\text{SEC}_{X/S}^d/\mathfrak{S}_d$ .*

*Proof.* We will construct canonical morphisms in both directions. Let  $(p : Z \rightarrow T, q : Z \rightarrow X)$  be a  $T$ -point of  $\mathcal{E}t_{X/S}^d$ . The scheme  $(Z/T)^d$  is étale of rank  $d^d$  over  $T$ . The diagonals of this scheme are open and closed, and their complement  $\text{SEC}_{Z/T}^d$  is étale of rank  $d!$ . This can be verified over algebraically closed points where it is trivial. The scheme  $\text{SEC}_{Z/T}^d$  comes with a  $\mathfrak{S}_d$ -action and a  $\mathfrak{S}_d$ -equivariant morphism  $\text{SEC}_{Z/T}^d \rightarrow (X/S)^d$ . This defines a  $T$ -point of  $[(X/S)^d/\mathfrak{S}_d]$ .

Conversely, let  $W \rightarrow T$  be a  $T$ -point of  $[(X/S)^d/\mathfrak{S}_d]$ , i.e., let  $W/T$  be a  $\mathfrak{S}_d$ -torsor together with a  $\mathfrak{S}_d$ -equivariant morphism  $W \rightarrow (X/S)^d$ . Let  $\mathfrak{S}_{d-1}$  be the subgroup of  $\mathfrak{S}_d$  acting by permuting the first  $d-1$  factors of  $(X/S)^d$ . This group acts freely on  $W$  and the quotient  $Z = W/\mathfrak{S}_{d-1}$  is an algebraic space, étale of rank  $d$  over  $T$ . Moreover, the composition of  $W \rightarrow (X/S)^d$  with the last projection is  $\mathfrak{S}_{d-1}$ -invariant and thus induces a morphism  $Z \rightarrow X$ . We have thus constructed a  $T$ -point of  $\mathcal{E}t_{X/S}^d$ .

It is clear that these constructions are functorial and thus defines morphisms  $F : \mathcal{E}t_{X/S}^d \rightarrow [(X/S)^d/\mathfrak{S}_d]$  and  $G : [(X/S)^d/\mathfrak{S}_d] \rightarrow \mathcal{E}t_{X/S}^d$ . It is not difficult to show that these are inverses. In fact, if  $(Z, p, q)$  is a  $T$ -point of  $\mathcal{E}t_{X/S}^d$ , then we have a canonical morphism  $\text{SEC}_{Z/T}^d/\mathfrak{S}_{d-1} \rightarrow Z$  and that this is an isomorphism can be checked over algebraically closed points. Conversely, if  $W/T$  is a  $\mathfrak{S}_d$ -torsor, then we obtain a morphism  $W \rightarrow ((W/\mathfrak{S}_{d-1})/T)^d$  where the  $i^{\text{th}}$  factor is given by composing the automorphism on  $W$  induced by the transposition  $\tau_{in} \in \mathfrak{S}_d$  with the quotient  $W \rightarrow W/\mathfrak{S}_{d-1}$ . Again, it is easily verified that  $W \rightarrow ((W/\mathfrak{S}_{d-1})/T)^d$  induces an isomorphism of  $W$  onto  $\text{SEC}_{(W/\mathfrak{S}_{d-1})/T}^d$ .  $\square$

**Remark (5.2).** The universal  $\mathfrak{S}_d$ -torsor of  $[(X/S)^d/\mathfrak{S}_d]$  is  $(X/S)^d$ . The above isomorphism shows that  $[(X/S)^d/\mathfrak{S}_{d-1}] = [(X/S)^{d-1}/\mathfrak{S}_{d-1}] \times_S X$  is the universal étale rank  $d$  family of  $\mathcal{E}t_{X/S}^d$ .



## 6. HILBERT SCHEMES AND HILBERT STACKS OF STACKS

The definitions of the Hilbert functor and Hilbert stack immediately generalize to the situation where  $X$  is an algebraic stack  $\mathcal{X}$ . For the Hilbert stack of points  $\mathcal{H}_{\mathcal{X}/S}^d$ , the objects are given by *schemes*  $Z$  with morphisms  $p$  and  $q$  as before. The morphisms are given by 2-cartesian diagrams and we only require that  $q_1$  and  $q_2 \circ \varphi$  agree up to a 2-morphism. For example, we have that  $\mathcal{H}_{\mathcal{X}/S}^1$  is the automorphism-free locus of  $\mathcal{X}$  and that  $\mathcal{H}_{\mathcal{X}/S}^1 = \mathcal{X}$ . If  $\mathcal{X}/S$  is separated then  $\mathcal{H}_{\mathcal{X}/S}^d$  is an open subfunctor of  $\mathcal{H}_{\mathcal{X}/S}^d$ .

It is clear that Proposition (2.4) and Corollaries (3.5), (3.8), and (4.7) remain valid for (*representable* in some instances) morphisms of arbitrary algebraic stacks  $f : \mathcal{X} \rightarrow \mathcal{Y}$ . We obtain the following generalization:

**Theorem (6.1).** *Let  $\mathcal{X}/S$  be an algebraic stack. Then  $\mathcal{H}_{\mathcal{X}/S}^d$  is algebraic. If  $\mathcal{X}/S$  has separated (resp. quasi-compact, resp. affine, resp. quasi-affine) diagonal, then so has  $\mathcal{H}_{\mathcal{X}/S}^d$ . If  $\mathcal{X}/S$  has one of the properties: quasi-compact, (locally) of finite presentation, (locally) of finite type; then so has  $\mathcal{H}_{\mathcal{X}/S}^d$ .*

*Proof.* Exactly as the proof of Theorem (4.5) but take a smooth presentation of  $\mathcal{X}$ .  $\square$

**Corollary (6.2).** *Let  $\mathcal{X}/S$  be a separated algebraic stack. Then  $\mathcal{H}_{\mathcal{X}/S}^d$  is represented by a separated algebraic space.*

*Proof.* As  $\mathcal{X}/S$  is separated, we have that  $\mathcal{H}_{\mathcal{X}/S}^d$  is an open subfunctor of  $\mathcal{H}_{\mathcal{X}/S}^d$  which is algebraic. The separatedness follows from Proposition (4.4).  $\square$

**Corollary (6.3).** *Let  $S' \rightarrow S$  be a finite flat morphism of rank  $d$ . Let  $\mathcal{X}'/S'$  be an, not necessarily separated, algebraic stack. Then the Weil restriction  $\mathbf{R}_{S'/S}(\mathcal{X}')$  is an algebraic stack. If  $\mathcal{X}'/S'$  has one of the properties of Proposition (2.4) then so has  $\mathbf{R}_{S'/S}(\mathcal{X}')/S$ . If the diagonal of  $\mathcal{X}'/S'$  has one of the properties (D), then so has the diagonal of  $\mathbf{R}_{S'/S}(\mathcal{X}')/S$ .*

*Proof.* The algebraicity of  $\mathbf{R}_{S'/S}(\mathcal{X}')$  follows immediately from Theorem (6.1) as  $\mathbf{R}_{S'/S}(\mathcal{X}')$  is an open substack of  $\mathcal{H}_{\mathcal{X}'/S'}^d$ . The properties of  $\mathbf{R}_{S'/S}(\mathcal{X}')$  and its diagonal follows from Remark (2.3) and Proposition (4.4).  $\square$

**Remark (6.4).** Note that even if  $\mathcal{X}'$  is separated, i.e., has proper diagonal, then  $\mathbf{R}_{S'/S}(\mathcal{X}')$  need not be separated unless  $S'/S$  is étale. For example, let  $S'/S$  be a finite flat ramified covering of degree  $d$  and let  $\mathcal{X}' = BG(S')$  where  $G$  is a finite group. Then  $\mathbf{R}_{S'/S}(\mathcal{X}')$  is a gerbe over  $S$  with generic geometric automorphism group  $G^d$  but with automorphism group of lower rank over the points of  $S$  where  $S'/S$  is ramified.

**Remark (6.5).** The argument in [Ols06, 3.3] shows that Corollary (6.3) remains true if  $S'/S$  is a proper quasi-finite flat *stack* admitting, fppf-locally on  $S$ , a finite flat presentation.

**Corollary (6.6).** *Let  $X \rightarrow S$  be a finite flat morphism of rank  $d$  (or an algebraic stack proper, quasi-finite and flat admitting fppf-locally a finite flat presentation) and  $\mathcal{Y} \rightarrow S$  an arbitrary algebraic stack. Then the stack  $\mathcal{H}om_S(X, \mathcal{Y})$  is algebraic. If the diagonal of  $\mathcal{Y}$  is a closed immersion (resp. unramified, resp. affine, resp. quasi-affine, resp. separated, resp. quasi-compact), then so is the diagonal of  $\mathcal{H}om_S(X, \mathcal{Y})$ .*

*Proof.* As  $\mathcal{H}om_S(X, \mathcal{Y}) = \mathbf{R}_{X/S}(X \times_S \mathcal{Y})$  this follows immediately from Corollary (6.3) (and Remark (6.5)).  $\square$

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